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Excitations in the dilute A_L lattice model: E_6, E_7 and E_8 mass spectra*

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Abstract. On the basis of features observed in the exact perturbation approach solution for the eigenspectrum of the dilute A_3 model, we propose expressions for excitations in the dilute A_4 and A_6 models. Principally, we require that these expressions satisfy the appropriate inversion relations. We demonstrate that they give the expected E_7 and E_6 mass spectra, and universal amplitudes, and agree with numerical expressions for the eigenvalues.

PACS. 05.70.Jk Critical point phenomena – 64.60.Cn Statistical mechanics of model systems – 75.10.Jm Quantized spin models

1 Introduction

The dilute A_L model is an exactly solvable, restricted solid-on-solid model defined on the square lattice. At criticality, the model can be constructed [1,2] from the dilute $O(n)$ loop model [3,4]. Each site of the lattice can take one of L possible (height) values, subject to the restriction that neighbouring sites of the lattice either have the same height, or differ by $\pm$ 1. Most importantly, the model can also be solved away from criticality. The off-critical Boltzmann weights of the allowed height configurations of an elementary face of the lattice are parametrised in terms of elliptic theta functions [1]. The interpretation of the elliptic nome p differs according to whether L is even or odd. In particular, for L odd the up-down symmetry of the Boltzmann weights is broken away from criticality. For $L = 3$ the elliptic nome plays the role of magnetic field. Moreover, the dilute A_3 model provides, in one of its regimes, an integrable lattice realisation of the E_8 Ising model, being in the same universality class as the two-dimensional Ising model in a magnetic field.

The calculation of the various off-critical thermodynamic properties of the model have verified this correspondence. The singular part of the bulk free energy of the dilute A_3 model in the appropriate regime gives the magnetic Ising exponent $\delta = 15$ [1], which also follows from the calculation of the local height probability [5]. The expected Ising magnetic surface exponent $\delta_s = -\frac{15}{7}$ follows from the excess surface free energy [6]. Moreover the E_8 mass spectrum,

$$
\begin{aligned}
m_2 &= 2\cos\frac{\pi}{5} &&= 1.618\,033\ldots \\
m_3 &= 2\cos\frac{\pi}{30} &&= 1.989\,043\ldots \\
m_4 &= 4\cos\frac{\pi}{5}\cos\frac{7\pi}{30} &&= 2.404\,867\ldots \\
m_5 &= 4\cos\frac{\pi}{5}\cos\frac{2\pi}{15} &&= 2.956\,295\ldots \\
m_6 &= 4\cos\frac{\pi}{5}\cos\frac{\pi}{30} &&= 3.218\,340\ldots \\
m_7 &= 8\cos^2\frac{\pi}{5}\cos\frac{7\pi}{30} &&= 3.891\,156\ldots \\
m_8 &= 8\cos^2\frac{\pi}{5}\cos\frac{2\pi}{15} &&= 4.783\,386\ldots
\end{aligned} \tag{1}
$$

predicted by Zamolodchikov [7,8] for the Ising model in a magnetic field is seen in the single particle excitation spectrum [9–12]. Here the masses are normalized such that $m_1 = 1$. They coincide with the components of the Perron-Frobenius vector of the Cartan matrix of the Lie algebra E_8.

In this paper we consider off-critical excitations in the dilute A_4 and A_6 models, which are expected to be related to the E_7 and E_6 scattering theories. The E_6 masses are

* Dedicated to J. Zittartz on the occasion of his 60th birthday

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(see, *e.g.*, [13–15] and references therein)

$$
\begin{aligned}
m_1 &= m_{\bar{1}} = 1 \\
m_2 &= 2\cos\frac{\pi}{4} &&= 1.414\ 213\ldots \\
m_3 &= m_{\bar{3}} = 2\cos\frac{\pi}{12} &&= 1.931\ 851\ldots \\
m_4 &= 4\cos\frac{\pi}{4}\cos\frac{\pi}{12} &&= 2.732\ 050\ldots
\end{aligned}
\tag{2}
$$

The E_7 masses, with $m_1 = 1$, are [13–15]

$$
\begin{aligned}
m_2 &= 2\cos\frac{5\pi}{18} &&= 1.285\ 575\ldots \\
m_3 &= 2\cos\frac{\pi}{9} &&= 1.879\ 385\ldots \\
m_4 &= 2\cos\frac{\pi}{18} &&= 1.969\ 615\ldots \\
m_5 &= 4\cos\frac{\pi}{18}\cos\frac{5\pi}{18} &&= 2.532\ 088\ldots \\
m_6 &= 4\cos\frac{\pi}{9}\cos\frac{2\pi}{9} &&= 2.879\ 385\ldots \\
m_7 &= 4\cos\frac{\pi}{18}\cos\frac{\pi}{9} &&= 3.701\ 666\ldots
\end{aligned}
\tag{3}
$$

Our approach begins in the next two sections by considering the inversion relations that hold for the off-critical dilute A_L models, how our solution [12] satisfies them in the case $L = 3$, and how the E_8 structure manifests itself within the solution. In the subsequent sections we propose solutions for A_4 and A_6 and demonstrate the expected E_7 and E_6 mass spectra. We conclude with some numerical evidence and discussion.

2 Inversion relations

The eigenvalues of the row transfer matrix of the dilute A_L model, defined on a periodic strip of width N, where we take N even, are [9]

$$
\begin{aligned}
\Lambda(u) = {}& \omega\left[\frac{\vartheta_1(2\lambda-u)\,\vartheta_1(3\lambda-u)}{\vartheta_1(2\lambda)\,\vartheta_1(3\lambda)}\right]^N \prod_{j=1}^{N}\frac{\vartheta_1(u-u_j+\lambda)}{\vartheta_1(u-u_j-\lambda)} \\
&+\left[\frac{\vartheta_1(u)\,\vartheta_1(3\lambda-u)}{\vartheta_1(2\lambda)\,\vartheta_1(3\lambda)}\right]^N \\
&\times\prod_{j=1}^{N}\frac{\vartheta_1(u-u_j)\,\vartheta_1(u-u_j-3\lambda)}{\vartheta_1(u-u_j-\lambda)\,\vartheta_1(u-u_j-2\lambda)} \\
&+\frac{1}{\omega}\left[\frac{\vartheta_1(u)\,\vartheta_1(\lambda-u)}{\vartheta_1(2\lambda)\,\vartheta_1(3\lambda)}\right]^N \prod_{j=1}^{N}\frac{\vartheta_1(u-u_j-4\lambda)}{\vartheta_1(u-u_j-2\lambda)},
\end{aligned}
\tag{4}
$$

where the N roots u_j are given by the Bethe equations

$$
\omega\left[\frac{\vartheta_1(\lambda-u_j)}{\vartheta_1(\lambda+u_j)}\right]^N = -\prod_{k=1}^{N}\frac{\vartheta_1(u_j-u_k-2\lambda)\,\vartheta_1(u_j-u_k+\lambda)}{\vartheta_1(u_j-u_k+2\lambda)\,\vartheta_1(u_j-u_k-\lambda)},
\tag{5}
$$

with $\omega = \exp(i\,\pi\ell/(L+1))$ for $\ell = 1, \ldots, L$. For regime 2, the regime to be considered, the spectral parameter u lies in the range $0 < u < 3\lambda$, with $\lambda = \pi s/r$, where $s = L+2$ and $r = 4(L+1)$.

The standard elliptic theta functions $\vartheta_1(u) = \vartheta_1(u,p)$ and $\vartheta_4(u) = \vartheta_4(u,p)$ of nome p are defined as

$$
\begin{aligned}
\vartheta_1(u) &= 2\,p^{\frac{1}{4}}\sin u\prod_{n=1}^{\infty}\left(1-2p^{2n}\cos 2u+p^{4n}\right)\left(1-p^{2n}\right), \\
\vartheta_4(u) &= \prod_{n=1}^{\infty}\left(1-2p^{2n-1}\cos 2u+p^{4n-2}\right)\left(1-p^{2n}\right).
\end{aligned}
\tag{6}
$$

Also of use are the conjugate variables

$$
w = e^{-2\pi u/\epsilon} \quad \text{and} \quad x = e^{-\pi^2/r\epsilon},
\tag{7}
$$

where nome $p = e^{-\epsilon}$. The relevant conjugate modulus transformations are

$$
\begin{aligned}
\vartheta_1(u,p) &= \left(\frac{\pi}{\epsilon}\right)^{\frac{1}{2}} e^{-(u-\pi/2)^2/\epsilon}\,E(w,q^2), \\
\vartheta_4(u,p) &= \left(\frac{\pi}{\epsilon}\right)^{\frac{1}{2}} e^{-(u-\pi/2)^2/\epsilon}\,E(-w,q^2),
\end{aligned}
\tag{8}
$$

where $q = e^{-\pi^2/\epsilon}$ and

$$
E(z,p) = \prod_{n=1}^{\infty}(1-p^{n-1}z)(1-p^n z^{-1})(1-p^n).
\tag{9}
$$

For this model, the partition function per site κ was first calculated using the inversion relation [1,5]

$$
\kappa(u)\,\kappa(u+3\lambda) = \frac{\vartheta_1(2\lambda-u)\vartheta_1(3\lambda-u)}{\vartheta_1^2(2\lambda)\vartheta_1^2(3\lambda)} \times\vartheta_1(2\lambda+u)\vartheta_1(3\lambda+u).
\tag{10}
$$

In this way the bulk free energy per site $f = \log\kappa$ was found to be

$$
f = \sum_{k=1}^{\infty}\left[\frac{(1-w^k)(1-x^{6sk}w^{-k})}{k(1-x^{2rk})(1+x^{6sk})} \times\ (x^{4sk}+x^{(2r-6s)k})(1+x^{2sk})\right].
\tag{11}
$$

The same result was derived [12,16] from the Bethe Ansatz solution for the groundstate eigenvalue $\Lambda_0(u)$.

Making use of the Poisson summation formula in the free energy (11) the leading singularity as $p \to 0$ in regime 2 is

$$
f \sim \mathcal{A}\,p^{r/3s},
\tag{12}
$$

where the amplitude $\mathcal{A}$ is given in terms of L by

$$\mathcal{A} = 4\sqrt{3}\,\frac{\cos\left[\dfrac{\pi(L+6)}{6(L+2)}\right]}{\sin\left[\dfrac{2\pi(L+1)}{3(L+2)}\right]}, \qquad (13)$$

and we have taken the isotropic value $u = 3\lambda/2$.

Excitations in the eigenspectrum can be considered in terms of the quantity

$$r_j(u) = \lim_{N\to\infty} \frac{\Lambda_j(u)}{\Lambda_0(u)}\,. \qquad (14)$$

The inversion relation (10) is simply

$$r_j(u)\, r_j(u+3\lambda) = 1, \qquad (15)$$

but there is a further relation to be satisfied [12],

$$r_j(u)\, r_j(u+2\lambda) = r_j(u+\lambda). \qquad (16)$$

Our approach here is not to solve the inversion relations directly, as was done, *e.g.*, by Klümper and Zittartz for the excitation spectra of the eight-vertex model [17]. Rather, in the light of our results for the excitations of the dilute A_3 model, we use the above inversion relations to give constraints on the Lie algebraic properties of a conjectured solution. We then test our results as best we can by numerically diagonalising the transfer matrix, and by comparison with results for E_7 and E_6 obtained by other methods.

3 The dilute A$_3$ model and the E$_8$ mass spectrum

We now summarise our results [12] for the dilute A_3 model, obtained by the exact perturbation approach [18]. The leading excitations in a given band of eigenvalues can be written in the compact form

$$r_j(w) = w^{n(a)} \prod_a \frac{E(-x^a/w)E(-x^{30-a}/w)}{E(-x^a w)E(-x^{30-a} w)}, \qquad (17)$$

where we have suppressed the nome x^{60} and the numbers a and $n(a)$ are given in Table 1. The E_8 numbers a have been discussed by McCoy and Orrick for the related Hamiltonian [11]. They appear, *e.g.*, in E_8 scattering theory [14] and in E_8 Lie algebraic polynomials [19]. The number $n(a)$ denotes the relevant band of eigenvalues.

Note that within a band of eigenvalues there may be more than one class of excitation. For example, in the leading band of eigenvalues there are two, which arise from a 2-string and a 4-string structure in the Bethe roots [9,10]. The expression (17) is the leading excitation for each class of eigenvalue. The last excitation within a class is also given by (17), but with positive argument in the elliptic functions.

Table 1. Parameters appearing in the eigenvalue expression (17).

j	$n(a)$	a
1	2	1, 11
2	2	7, 13
3	3	2, 10, 12
4	3	6, 10, 14
5	4	3, 9, 11, 13
6	4	6, 8, 12, 14
7	5	4, 8, 10, 12, 14
8	6	5, 7, 9, 11, 13, 15

In the original variables (17) reads

$$r_j(u) = \prod_a \frac{\vartheta_4\left(\dfrac{a\pi}{60} - \dfrac{8u}{15}\right)\vartheta_4\left(\dfrac{(30-a)\pi}{60} - \dfrac{8u}{15}\right)}{\vartheta_4\left(\dfrac{a\pi}{60} + \dfrac{8u}{15}\right)\vartheta_4\left(\dfrac{(30-a)\pi}{60} + \dfrac{8u}{15}\right)} \qquad (18)$$

with nome $p^{8/15}$.

The various correlation lengths follow as

$$\xi_j^{-1} = -\log r_j(u), \qquad (19)$$

where we take the relevant leading eigenvalue at the isotropic point $u = 3\lambda/2$, which for $L = 3$ is $u = 15\pi/32$.

The fundamental correlation lengths can thus be written

$$\begin{aligned} m_j = \xi_j^{-1} &= \sum_a \log \frac{\vartheta_4\left(\dfrac{a\pi}{60} + \dfrac{\pi}{4}\right)\vartheta_4\left(\dfrac{(30-a)\pi}{60} + \dfrac{\pi}{4}\right)}{\vartheta_4\left(\dfrac{a\pi}{60} - \dfrac{\pi}{4}\right)\vartheta_4\left(\dfrac{(30-a)\pi}{60} - \dfrac{\pi}{4}\right)} \\ &= 2\sum_a \log \frac{\vartheta_4\left(\dfrac{a\pi}{60} + \dfrac{\pi}{4}\right)}{\vartheta_4\left(\dfrac{a\pi}{60} - \dfrac{\pi}{4}\right)}. \end{aligned} \qquad (20)$$

Expanding this expression in powers of p gives

$$m_j \sim 8\, p^{8/15} \sum_a \sin\frac{a\pi}{30} \quad \text{as} \quad p \to 0\,. \qquad (21)$$

This is the formula obtained by McCoy and Orrick [11] for the Hamiltonian, from which the E_8 masses in (1) are recovered by virtue of trig identities.

In particular,

$$\xi_1^{-1} \sim 8\, p^{8/15}\left(\sin\frac{\pi}{30} + \sin\frac{11\pi}{30}\right) = 16 \sin\frac{\pi}{5}\cos\frac{\pi}{6}\; p^{8/15}. \qquad (22)$$

We are now able to consider the universal magnetic Ising amplitude [16]. From (12, 13),

$$f \sim 4\sqrt{3}\,\frac{\sin\dfrac{\pi}{5}}{\cos\dfrac{\pi}{30}}\, p^{16/15} \quad \text{as} \quad p \to 0\,. \qquad (23)$$

Combining this with (22) gives

$$f\,\xi_1^2 = \frac{1}{16\sqrt{3}\sin\dfrac{\pi}{5}\cos\dfrac{\pi}{30}} = 0.061\ 728\ 589\ldots \qquad (24)$$

as $p \to 0$. This is the result for the universal magnetic Ising amplitude obtained earlier by thermodynamic Bethe Ansatz calculations based on the E_8 scattering theory [15] (see also Ref. [20] in the context of the form-factor bootstrap approach). Here it has been obtained from the lattice model.

From the outset, no assumptions were made on the E_8 structure in the dilute A_3 model, both in our own calculations, and in the thermodynamic Bethe Ansatz calculations [9,11]. We now highlight a few of the E_8 features as a guide to our considerations of E_7 and E_6.

First, each a value occurs in (17) together with its complement in (30), the Coxeter number of E_8, so that no integer greater than 15 appears in the lists in Table 1.

Second, the inversion relation

$$r_j(w)\,r_j(x^{30}w) = 1, \qquad (25)$$

is satisfied by an expression of the form (17) for any a value. However, the stronger relation

$$r_j(w)\,r_j(x^{20}w) = r_j(x^{10}w), \qquad (26)$$

is satisfied if, within the set of integers, one finds not only a, where $a = 1, ..., 9$, but also $a + 10$, or equivalently its complement in 30, $20 - a$, by virtue of the properties

$$E(z,p) = E(p/z,p) = -zE(z^{-1},p). \qquad (27)$$

The integer $a = 10$ may appear alone, because the factor it contributes to $r_j(w)$ satisfies (26) by itself. From Table 1, the sets of integers found by the perturbative approach [12] all have these features.

Finally, we observe that the nome p cancels in (24) because of the relationship between the power of p occurring in f and in ξ_1. Indeed, this combination defines the hyperscaling relation between the corresponding critical exponents.

4 The dilute A_4 model and the E_7 mass spectrum

We now use our observations for E_8 to arrive at a conjecture (equivalent to (17)) for the excitations of E_7.

The free energy expression is, from (12, 13),

$$f \sim \frac{2\sqrt{3}}{\sin\dfrac{5\pi}{18}}\,p^{10/9} \quad \text{as} \quad p \to 0\,. \qquad (28)$$

In order to obtain a finite expression from $f\,\xi_1^2$, we thus expect

$$\xi_1^{-1} \sim p^{5/9} \quad \text{as} \quad p \to 0\,. \qquad (29)$$

This power of the nome must appear in the expression equivalent to (18) for E_7, and is thus related to the one we must propose for $r_j(w)$ by the conjugate modulus transformation (8), namely

$$e^{-5\epsilon/9} \to e^{-18\pi^2/5\epsilon} = x^{72}, \qquad (30)$$

where for $L = 4$, $x = e^{-\pi^2/20\epsilon}$.

The inversion relation in conjugate modulus form is

$$r_j(w)\,r_j(x^{36}w) = 1. \qquad (31)$$

However, the Coxeter number for E_7 is 18, so that we expect to select our integers from 1, ..., 9. We thus propose that the excitations for E_7 take the form

$$r_j(w) = w^{n(a)} \prod_a \frac{E(-x^{2a}/w)E(-x^{36-2a}/w)}{E(-x^{2a}w)E(-x^{36-2a}w)} \qquad (32)$$

with nome x^{72}. The additional relation which serves to constrain the possible a values is

$$r_j(w)\,r_j(x^{24}w) = r_j(x^{12}w). \qquad (33)$$

This condition is satisfied if, within a set of possible integers, a appears together with $a+6$ or equivalently $12-a$, apart from $a = 6$ whose contribution satisfies (33) by itself.

Written in terms of the original variables the expression (32) is

$$r_j(u) = \prod_a \frac{\vartheta_4\left(\dfrac{a\pi}{36} - \dfrac{5u}{9}\right)\vartheta_4\left(\dfrac{(18-a)\pi}{36} - \dfrac{5u}{9}\right)}{\vartheta_4\left(\dfrac{a\pi}{36} + \dfrac{5u}{9}\right)\vartheta_4\left(\dfrac{(18-a)\pi}{36} + \dfrac{5u}{9}\right)} \qquad (34)$$

with nome $p^{5/9}$. Taking the isotropic value $u = 9\pi/20$ we obtain

$$m_j = \xi_j^{-1} = 2\sum_a \log \frac{\vartheta_4\left(\dfrac{a\pi}{36} + \dfrac{\pi}{4}\right)}{\vartheta_4\left(\dfrac{a\pi}{36} - \dfrac{\pi}{4}\right)} \qquad (35)$$

for the masses, and so

$$m_j \sim 8\,p^{5/9} \sum_a \sin\frac{a\pi}{18} \quad \text{as} \quad p \to 0\,. \qquad (36)$$

We now turn to the sets of integers associated with E_7 in the context of Lie algebraic polynomials [19] which form the first six rows of the third column of Table 2. Clearly these integers satisfy the constraints described above as being placed upon them by (33). Together with the last row, they are also to be found within the table given for E_7 scattering in [14].

Table 2. Parameters appearing in the eigenvalue expression (32).

j	$n(a)$	a
1	1	6
2	2	1, 7
3	2	4, 8
4	2	5, 7
5	3	2, 6, 8
6	3	4, 6, 8
7	4	3, 5, 7, 9

Applying trig identities to the sum in (36) with these sets of integers gives

$$\begin{aligned}
\sum_{a=6} \sin\frac{a\pi}{18} &= \sqrt{3}/2,\\
\sum_{a=1,7} \sin\frac{a\pi}{18} &= \sqrt{3}\cos\frac{5\pi}{18},\\
\sum_{a=4,8} \sin\frac{a\pi}{18} &= \sqrt{3}\cos\frac{\pi}{9},\\
\sum_{a=5,7} \sin\frac{a\pi}{18} &= \sqrt{3}\cos\frac{\pi}{18},\\
\sum_{a=2,6,8} \sin\frac{a\pi}{18} &= 2\sqrt{3}\cos\frac{\pi}{18}\cos\frac{5\pi}{18},\\
\sum_{a=4,6,8} \sin\frac{a\pi}{18} &= 2\sqrt{3}\cos\frac{\pi}{9}\cos\frac{2\pi}{9},\\
\sum_{a=3,5,7,9} \sin\frac{a\pi}{18} &= 2\sqrt{3}\cos\frac{\pi}{18}\cos\frac{\pi}{9},
\end{aligned} \tag{37}$$

which, apart from normalisation, correspond to m_1, ..., m_7 of (3)[1]. As another piece of evidence for our identification of $a = 6$ with m_1, from which the others follow, we consider the amplitude

$$f\,\xi_1^2 = \frac{2\sqrt{3}}{\sin\dfrac{5\pi}{18}}\frac{1}{(8\sin\dfrac{\pi}{3})^2} = \frac{1}{8\sqrt{3}\cos\dfrac{2\pi}{9}} \tag{38}$$

as $p \to 0$. This is in agreement with the E_7 thermodynamic Bethe Ansatz result [15].

[1] There is another relationship between the E_7 mass ratios, the trigonometric expression of (36) and integers in the table of [14]. However, the one described here is necessary in the context of the solvable dilute A_4 model in order to satisfy its inversion relations.

5 The dilute A_6 model and the E_6 mass spectrum

The free energy expression for the dilute A_6 model is, again from (12, 13),

$$f \sim \frac{2\sqrt{6}}{\cos\dfrac{\pi}{12}}\, p^{7/6} \quad \text{as} \quad p \to 0\,, \tag{39}$$

and so we expect

$$\xi_1^{-1} \sim p^{7/12} \quad \text{as} \quad p \to 0\,. \tag{40}$$

This power of the nome must appear in the expression equivalent to (18) for E_6, and thus prescribes the nome of the expression we propose for $r_j(w)$, because in the conjugate modulus transformation (8),

$$e^{-7\epsilon/12} \to e^{-24\pi^2/7\epsilon} = x^{96}, \tag{41}$$

where in the case $L = 6$, $x = e^{-\pi^2/28\epsilon}$. The inversion relation in conjugate modulus form is

$$r_j(w)\, r_j(x^{48}w) = 1. \tag{42}$$

Finally, the Coxeter number for E_6 is 12, so that we expect to select our integers from 1, ..., 6. We thus propose that the excitations for E_6 take the form

$$r_j(w) = w^{n(a)} \prod_a \frac{E(-x^{4a}/w)E(-x^{48-4a}/w)}{E(-x^{4a}w)E(-x^{48-4a}w)} \tag{43}$$

with nome x^{96}.

The additional relation which serves to constrain the possible a values is

$$r_j(w)\, r_j(x^{32}w) = r_j(x^{16}w). \tag{44}$$

Thus within any set of possible integers, a must appear together with $a + 4$ or equivalently $8 - a$, apart from $a = 4$ which satisfies (44) by itself. Written in terms of the original variables the expression (43) is

$$r_j(u) = \prod_a \frac{\vartheta_4\left(\dfrac{a\pi}{24} - \dfrac{7u}{12}\right)\vartheta_4\left(\dfrac{(12-a)\pi}{24} - \dfrac{7u}{12}\right)}{\vartheta_4\left(\dfrac{a\pi}{24} + \dfrac{7u}{12}\right)\vartheta_4\left(\dfrac{(12-a)\pi}{24} + \dfrac{7u}{12}\right)} \tag{45}$$

with nome $p^{7/12}$. Taking the isotropic value $u = 3\pi/7$ we obtain

$$m_j = \xi_j^{-1} = 2\sum_a \log\frac{\vartheta_4\left(\dfrac{a\pi}{24} + \dfrac{\pi}{4}\right)}{\vartheta_4\left(\dfrac{a\pi}{24} - \dfrac{\pi}{4}\right)} \tag{46}$$

for the masses. Thus

$$m_j \sim 8\,p^{7/12} \sum_a \sin\frac{a\pi}{12} \quad \text{as} \quad p \to 0\,. \tag{47}$$

Table 3. Parameters appearing in the eigenvalue expression (43).

j	$n(a)$	a
1 ,$\bar{1}$	1	4
2	2	1, 5
3, $\bar{3}$	2	3, 5
4	3	2, 4, 6

The integers given in Table 3 satisfy the constraint placed upon them by (44). Apart from the entry for $j = 4$, these integers are again to be found in [19], and they appear within the table of [14] for E_6.

Applying trig identities to the sum in (47) with these sets of integers gives

$$\begin{aligned} \sum_{a=4} \sin \frac{a\pi}{12} &= \sqrt{3}/2, \\ \sum_{a=1,5} \sin \frac{a\pi}{12} &= \sqrt{3}/\sqrt{2}, \\ \sum_{a=3,5} \sin \frac{a\pi}{12} &= \sqrt{3} \cos \frac{\pi}{12}, \\ \sum_{a=2,4,6} \sin \frac{a\pi}{12} &= \sqrt{6} \cos \frac{\pi}{12}, \end{aligned} \tag{48}$$

which, apart from normalisation, correspond to $m_1, \ldots, m_4$ of (2). Our identification of $a = 4$ with m_1, gives the amplitude

$$f\, \xi_1^2 = \frac{2\sqrt{6}}{\cos \frac{\pi}{12}} \frac{1}{(4\sqrt{3})^2} = \frac{1}{2\sqrt{3}(1+\sqrt{3})} \tag{49}$$

as $p \to 0$, which is in agreement with the thermodynamic Bethe Ansatz result [15].

6 Numerical evidence and discussion

Based on our result (17) for the eigenspectrum of the dilute A_3 lattice model in regime 2, and its resulting E_8 structure, we have proposed analogous formulae for the dilute A_4 and A_6 models under the assumption of corresponding E_7 and E_6 structures. Such correspondence is to be expected on a number of grounds. For example, at criticality the central charges of the dilute A_L models are known from the underlying loop model [1]. In regime 2, $c = 7/10$ for the A_4 model and $c = 6/7$ for the A_6 model. These are the same as the E_7 and E_6 values [13].

A number of considerations have motivated our final results. Our first input was the hyperscaling relation, $f\xi^2 =$ constant, which constrains the power of the elliptic nome p appearing in the inverse correlation lengths. We found that the stronger inversion relation (16) constrains the set of integers a appearing in the eigenvalue expressions. We took these numbers from the Lie algebraic polynomials tabulated by Kostant [19]. Our results produce the E_6 (2) and E_7 (3) masses in the critical limit $p \to 0$. However, the configuration of a's for the heaviest mass does not appear in the Kostant polynomials. We chose that configuration to be consistent with the predicted E_6 and E_7 mass spectra, and subsequently noted that it had been observed in the context of scattering theory [14]. Our identification of the a's associated with the lightest masses also gives the universal amplitudes (49, 38) in agreement with the thermodynamic Bethe Ansatz results based on the E_6 and E_7 algebras [15].

Table 4. Numerical estimates with increasing system size N of leading eigenvalue ratios in the dilute A_6 model at $\lambda = 2\pi/7$. Also shown is the expected exact result (50) in the thermodynamic limit. The corresponding values of a are as given in Table 3.

	N	Λ_0/Λ_1	Λ_0/Λ_2	Λ_0/Λ_3
$p = 0.1$	3	6.0279		
	4	6.7882		
	5	6.9281		
	6	6.9474	15.268	
	7	6.9501	15.511	41.05
	∞	6.9505	15.590	42.44
$p = 0.3$	3	89.93047		
	4	90.08438		
	5	90.08605	652.6278	
	6	90.08607	652.7399	6434.75
	7	90.08607	652.7442	6436.87
	∞	90.08607	652.7444	6437.08

We have performed a number of numerical tests on the eigenspectra of the dilute A_4 and A_6 models to check our results. Specifically, we have diagonalised the periodic row-transfer matrix for finite lattice sizes. Consider the dilute A_6 model first. Here $\lambda = 2\pi/7$. The largest eigenvalue Λ_0 is 3-fold degenerate in the thermodynamic limit. Successive numerical estimates of the first few eigenvalue ratios Λ_0/Λ_j at the isotropic point $u = 3\lambda/2$ are tabulated in Table 4 for the values $p = 0.1$ and $p = 0.3$. Excellent agreement is seen with the expected result (45), which reduces to

$$\frac{\Lambda_0}{\Lambda_j} = \prod_a \left[\frac{\vartheta_4\left(\frac{a\pi}{24} + \frac{\pi}{4}, p^{7/12}\right)}{\vartheta_4\left(\frac{a\pi}{24} - \frac{\pi}{4}, p^{7/12}\right)} \right]^2. \tag{50}$$

The dilute A_4 model at $\lambda = 3\pi/10$ is more complicated. A detailed numerical study of the Bethe Ansatz equations has revealed all seven masses [21]. However, the eigenvalue spectrum is dependent on the sign of p. In this case, all of the E_7 masses are observed in the $p < 0$ regime (regime 2^-). Only a subset is observed for $p > 0$ (regime 2^+). Our numerical results for the first few leading eigenvalues are shown in Table 5 for $p = -0.3$. The eigenvalues Λ_1 and Λ_3 do not appear in the eigenspectrum for $p = 0.3$. Clearly there is excellent agreement with our result (34),

Table 5. Numerical estimates with increasing system size N of leading eigenvalue ratios in the dilute A_4 model at $\lambda = 3\pi/10$. Also shown is the expected exact result (51) in the thermodynamic limit. The corresponding values of a are as given in Table 2.

	N	Λ_0/Λ_1	Λ_0/Λ_2	Λ_0/Λ_3	Λ_0/Λ_4
$p = -0.3$	4	116.09490	492.5475		
	5	116.09969	493.2263	8669.13	
	6	116.09973	493.2413	8724.17	11928
	7	116.09973	493.2416	8726.53	12067
	∞	116.09973	493.2416	8726.64	12190

which here simplifies to

$$\frac{\Lambda_0}{\Lambda_j} = \prod_a \left[\frac{\vartheta_4\left(\frac{a\pi}{36} + \frac{\pi}{4}, p^{5/9}\right)}{\vartheta_4\left(\frac{a\pi}{36} - \frac{\pi}{4}, p^{5/9}\right)} \right]^2 . \tag{51}$$

We expect this result to hold in regime 2^- for all of the masses, or correspondingly for each set of a's given in Table 2. Apart from Λ_1 and Λ_3, we have not explored further which of the eigenvalues are absent in regime 2^+. We await the publication of reference [21].

In contrast with the dilute A_4 model, the mass spectrum of the dilute A_6 model appears to be equivalent in regimes $2^\pm$. Such equivalence holds for the dilute A_L models with L odd, where the eigenspectrum is independent of the sign of p. This is a consequence of the off-critical weights breaking the Z_2 symmetry for L odd. However, for L even this symmetry is not broken. As to why the mass spectrum may be the same for the dilute A_6 model in regimes $2^\pm$, this remains one of the mysteries of the dilute A_L models for L even, which are yet to be investigated.

Finally we note that although the evidence for our conjectured results is convincing, they of course await a formal derivation.

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